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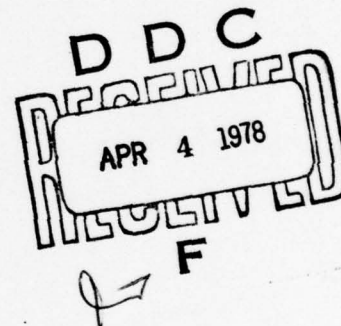
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# ELECTRIC CONDUCTIVITY IN A BEAM, PLASMA SYSTEM

BY RONALD L. KLIGMAN

RESEARCH AND TECHNOLOGY DEPARTMENT

15 SEPTEMBER 1977



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SUMMARY

The electric conductivity tensor of a beam-plasma interacting system is an extremely significant operator. Within the model we have chosen, one calculates the effects of the beam on the plasma as corrections to the classical results found from Drude theory. This work has been supported by the Independent Research Program of the Center.

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## I. INTRODUCTION

Many approaches to the understanding of the frequency-dependent, electrical conductivity tensor of a plasma have been made over the years. Depending on the plasma model one was considering, investigators have chosen the relevant transport equation that best suited their needs. One of the earliest treatments of the conductivity was that done by Drude.<sup>1</sup> He considered a gas of electrons that were free to move and collide with a collection of fixed ions at rest. When an external electric field is applied the motion of individual electrons is governed by Newton's law which contains a field-dependent term and a collision frequency term. The resulting conductivity depends inversely on the collision frequency in the low frequency limit of the applied electric field. This is the so-called DC conductivity. Since the average collision frequency is proportional to the mean velocity of the electron, the conductivity goes inversely as the square root of the absolute temperature of the electrons. Cohen, Spitzer and Routley<sup>2</sup> used the Boltzmann equation to obtain the velocity distribution function for an electron gas mixed with singly-ionized atoms. They treated the collision term (i.e. the time rate of change in the distribution function due to collisional processes) as composed of two parts. The first part is the traditional representation<sup>3</sup> for close encounters which produce large deflections in particle trajectories. The second represents a diffusion process, such as one finds in Brownian motion. Thus distant encounters produce only minor deflections in the motion of particles. The DC conductivity obtained from

this theory exhibits a temperature behavior that is proportional to the  $3/2$  power. In a later paper, Spitzer and Harm<sup>4</sup> included the effects of electron-electron interactions on the collision term represented by the Fokker-Planck equation, for a fully ionized gas. The results were quantitatively comparable to those of Chapman and Cowling<sup>3</sup> and Cowling.<sup>5</sup> Jackson<sup>6</sup> was able to combine the Drude theory with an expression for the mean ion-electron collision frequency. This expression results from a careful treatment of the multiple scattering of electrons as they pass through a finite region of matter. His results compare with those of Spitzer to within a factor of 2. Paramount to this result was the assumption of small-angle Coulomb scattering as the dominant process. These results are, of course, for the zero frequency limit (or DC) conductivity. Rosenbluth, MacDonald and Judd<sup>7</sup> found a series representation for the distribution function satisfying the Fokker-Planck equation. The series is given in terms of Legendre functions relative to some axis of symmetry that is specified by an external field. The first term in the series corresponds to Chandrashekhar's<sup>8</sup> calculation, the second, to that of Cohen, Spitzer and Routly.<sup>2</sup> Retaining higher order terms in the distribution function expansion will lead to a temperature dependence in the DC electric conductivity that departs from the  $3/2$  power behavior. These approaches to the electrical conductivity do not retain the dynamics of the transport process itself except for the very simplistic Drude theory. The response to some time-dependent, external electric field, in general, will depend on the history of the disturbance as well as

its present value. For a plasma medium that is subjected to a time-varying external electric field, the medium takes a finite amount of time to respond to the field and cannot keep up with its changes. Thus the electric polarization of the medium at any given time will be responding to excitations experienced at some time in the past. More concisely, the electric displacement and the electric field will be functionally related through a convolution in time.<sup>9,10</sup> Thus the electric conductivity and the electric permeability of the medium are frequency-dependent quantities.

An alternative approach to the study of such time-dependent phenomena is a theory of irreversible processes for a system that is not very far from thermodynamic equilibrium. This theory relates the thermodynamic average of the time fluctuations of generalized forces which act on a system to some dissipation parameter or kinetic coefficient. For the situation at hand, the generalized force would be the applied, time-dependent electric field, the kinetic coefficient would be given by the electric conductivity. Nyquist<sup>11</sup> introduced the notion of coupling between the voltage fluctuations in a linear electrical network to the electrical resistance of the system. The dissipation process is the result of the system attempting to come into thermal equilibrium with the external source. The fluctuations in the external source serve as the mechanism for the transfer of energy to the dissipative, or electrical system, and dissipation results from the randomizing effect of the fluctuations on the external source. Kirkwood<sup>12</sup> found an expression for the friction constant that

plays an important role in the theory of Brownian motion through the Langevin equation. The correlation of intermolecular forces acting on a given particle in some system of particles is proportional to the friction constant. Callen and Welton<sup>13</sup> generalized Nyquist's results and obtained a relationship between the general impedance of a linear system and fluctuations of generalized forces. The time correlations between a generalized force and any function of momenta and coordinates for a statistical system was considered by Takahasi.<sup>14</sup> His results served to unify the static statistical theory of Gibbs<sup>15</sup> to the time dependent phenomena represented by thermal noise in electrical circuits, as discussed by Nyquist.<sup>11</sup>

The first attempt to apply the general concept of the fluctuation dissipation theorem to the problem of electric conduction in metals was made by Kubo and co-workers.<sup>16</sup> They obtained expressions for the generalized susceptibilities for systems that are strongly coupled to external generalized forces, and related these susceptibilities to fluctuations in observable macroscopic quantities such as particle coordinates or momenta. In the case of the conduction problem, the applied electric field plays the role of the generalized force. H. Mori<sup>17</sup> successfully unified Kirkwood's theory<sup>12</sup> of statistical mechanics with the stochastic theory of Brownian motion embodied by the Langevin equation.<sup>8</sup> The resulting generalized Langevin equation successfully treats the behavior of transport processes through the time evolution of microscopic internal processes such as the temperature gradient and stress tensor in fluids. He also distinguishes between two

types of relaxation processes in fluids, a macroscopic process which is represented by hydrodynamic equations and a microscopic process which allows for local thermodynamic equilibrium. The electric conduction problem studied by Kubo<sup>16</sup> is analogous to the macroscopic process in fluids studied by Mori.<sup>17</sup> A study of plasma transport coefficients using the correlation function approach of Kubo and Mori was made by Bartis and Oppenheim.<sup>18</sup> In particular, the self-diffusion coefficient was studied in the absence of external electric or magnetic fields. Short range forces, as represented by a binary collision operation, and long range Coulombic forces are used to represent the forces acting on the particles in the system. Diagrammatic techniques are employed to conveniently sum the non-zero contributions to the interactions and to arrive at convergent values.

Questions have been raised concerning the failure of Kubo's theory to distinguish between the observed, macroscopic field, and the applied field. Izuyama<sup>19</sup> included the effects of Coulomb interaction between the electrons in the metal as well as the interaction of the electrons with the external field. His spatially-dependent, and frequency-dependent conductivity has as its leading term, the Kubo result.<sup>16</sup> Higher order terms correspond to the Coulomb interaction. He obtains an expression for the conductivity that represents the ratio of the current responding to the effective field in the metal to the macroscopic, effective field. This effective field is nothing more than the applied field plus the induced polarization field.

We shall use the correlation function method developed by

Kubo<sup>16</sup> to study the electric conductivity of a plasma in an external field. The origin of the field will be a line source of charge and current, which serves to model an electron beam. The time evolution operator for the current flowing in the plasma shall be represented by the usual one established by Kirkwood,<sup>12</sup> the harmonic, exponential, Liouville operator. A mean field theory approximation will be introduced to simplify the expressions for the interactions between the various charged particles in the plasma. The time dependent response function is expressed in terms of the radial and axial current components within the plasma. The Fourier transform of this quantity directly yields the tensor components of the frequency dependent electric conductivity. Ignoring spatial changes in the Hamiltonian yields the usual results of Drude theory, retention of the changes introduces some additional complexity in the frequency dependence of the conductivity.

We shall postpone for a later paper the effects of binary collisions on the conductivity tensor, where we shall study the "nonlocal" characteristics of the conductivity as well.

## II. THE ELECTRIC CONDUCTIVITY, THE ADMITTANCE FUNCTION AND THE RESPONSE FUNCTION

The introduction of a relativistic electron beam into a neutral gas disturbs the initial state of the system quite abruptly and dramatically. If, however, instead of a sharp "switching on" of the beam, we adiabatically insert the beam into the gas before we define the initial state of the system and the beam particle density is much less than the gas density, then the beam-gas interaction may be represented by some perturbation energy. A simplified model of this system shall be a plasma (the result of ionization by the beam front) in the presence of an external current, charge source of infinite length which is the initial state of the system at  $t = -\infty$ . The details of the ionization process and other attendant phenomena such as recombination, excitation, and radiative processes shall be totally ignored. What shall be addressed in this paper is the response of the plasma to the external field. A brief discussion relating the generated conductivity to the general admittance and response function will help to establish the approach to charge conduction.

If we represent the induced time-dependent current as a change in some initial value we can write

$$\Delta J(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \tilde{J}(\omega) \quad (2.1)$$

We have not allowed for spatial variation in the current. This assumption is reasonable if we assert that the external electromagnetic field varies quite slowly through space, over distances large compared to the Debye length of the plasma. Furthermore, for simplicity we restrict ourselves to one dimension so that we

may most easily establish relations that can be directly extended to the three dimensional case. For convenience, we also assume there is one Fourier component that dominates heavily over all others, then we may write approximately,

$$E(t) \sim E_0 e^{i\omega t} \quad (2.2)$$

where  $E(t)$  is the external, perturbing electric field.

We use the generalization of Ohm's law which is valid for rapidly time varying fields whose principle frequencies are not small in comparison to those frequencies that characterize the electric properties of the plasma, such as the plasma frequency. The generalization is

$$\bar{J}(\omega) = \sigma(\omega) \bar{E}(\omega) \quad (2.3)$$

and the frequency dependent conductivity  $\sigma$  is a scalar in the one dimensional case.

We then obtain the result

$$\Delta J(t) = \text{Re}\{E_0 \sigma(\omega) e^{i\omega t}\} \quad (2.4)$$

This equation relates the response  $\Delta J(t)$  to the excitation  $E_0 e^{i\omega t}$  directly, and the conductivity may be interpreted as a general admittance function which has been well described by many researchers.<sup>11,13,14,16</sup> In particular, Kubo<sup>16</sup> has related the general admittance function to the response function, as Fourier transforms of each other. Using his results we may write

$$\Delta J(t) = \int_{-\infty}^t \hat{\sigma}(t - t') E(t') dt' \quad (2.5)$$

$\hat{\sigma}$  is the response function and eq. (2.5) says that the sum of excitations that act on the system from the present time back through infinite past times produces a net response  $\Delta J(t)$ , the current that flows in the plasma. Kubo derived an expression for the response function in terms of an equilibrium ensemble average of a product of current components, namely

$$\hat{\sigma}(t) = \frac{1}{n k T} \langle \Delta J(0) \Delta J(t) \rangle_0 \quad (2.6)$$

This expression assumes the plasma to be a collisionless, single component system of volume charge density  $n$  comprised solely of electrons having charge  $e$ . The angular bracket represents an equilibrium ensemble average. This is an extremely important point in the interpretation of this formalism. The presence of the external electric field does not push the initial system far from equilibrium which allows us to express any change in the distribution function as a linear functional of the equilibrium distribution function. This obviates the need to solve some difficult transport equation for the non-equilibrium distribution function, and allows us to substitute an easier problem, that of the calculation of the current correlation function, which represents the effect of a perturbation on the originally quiescent system. For a three dimensional system the frequency dependent conductivity tensor is given by the Fourier transform of eq. (2.6),

$$\sigma_{\mu\nu}(\omega) = \frac{ne^2}{m^2 k T} \lim_{\epsilon \rightarrow 0} \int_0^\infty dt e^{-(\epsilon + i\omega)t} \langle p_\mu(0) p_\nu(t) \rangle_0 \quad (2.7)$$

We have recast the correlation function in terms of the charged particles momentum components rather than the current. The exponential damping factor is a convergence factor that eliminates any persistent oscillations that may be present in the response function. The static conductivity is the zero frequency limit of eq. (2.7), i.e.

$$\sigma_{\mu\nu}(0) = \frac{ne^2}{m kT} \lim_{\epsilon \rightarrow 0} \int_0^{\infty} dt e^{-\epsilon t} \langle p_{\mu}(0) p_{\nu}(t) \rangle_0 \quad (2.8)$$

A further comment on eqs. (2.6) through (2.8) is required to emphasize that these expressions result from a linear approximation to the Poisson bracket of the distribution function and the perturbation. Retaining higher order, non-linear terms would include higher inverse powers of  $kT$  and stronger time dependence. This will consequently introduce higher inverse powers of frequency in the frequency dependent conductivity tensor. Thus the results we shall present in this paper are regarded as a linear approximation which may require further refinement in the future.

## III. THE HAMILTONIAN

We shall consider the electrons in the plasma as the primary carriers of current, and shall ignore electron-electron collisions, and electron-ion collisions. The Hamiltonian for such a system of  $N$  electrons in the presence of an external line charge and current source may be written

$$H = \frac{1}{2m} \sum_{j=1}^N \left[ \vec{p}_j - e \vec{A}(\vec{r}_j) \right] \cdot \left[ \vec{p}_j - e \vec{A}(\vec{r}_j) \right] + e \sum_{j=1}^N \phi(r_j) \quad (3.1)$$

$\vec{A}(\vec{r}_j)$  and  $\phi(\vec{r}_j)$  are the vector and scalar potentials associated with the external sources and the electrons in the plasma. Let us assume there is a uniform distribution of charge in the  $z$  direction. Then we have the scalar potential expressed as a superposition of plasma sources and external sources,

$$\phi(\vec{r}) = \phi_0(r) + \sum_{j=1}^N \phi_j(\vec{r}, \vec{r}_j) \quad (3.2)$$

where

$$\nabla^2 \phi_0(r) = - \frac{\lambda}{\epsilon_0} \frac{\delta(r)}{r} \quad (3.3)$$

and

$$\nabla^2 \phi_j(r, \theta; r_j, \theta_j) = \frac{e'}{\epsilon_0} \frac{\delta(r-r_j)}{r} \delta(\theta-\theta_j) \quad (3.4)$$

The potentials satisfy Poisson's equation with  $\lambda$  and  $e'$  representing

the charge per unit length, i.e.  $dq/dz$  for the external source and the plasma, respectively. The variables  $r$  and  $r_j$  represent the field and source radial coordinates respectively. The solutions to eqs. (3.2) and (3.3) are<sup>6</sup>

$$\phi_o(r) = -\frac{\lambda}{2\pi\epsilon_o} \ln(r/R) \quad (3.5)$$

$$\phi_j(r, \theta; r_j, \theta_j) = \begin{cases} -\frac{e'}{2\pi\epsilon_o} \left[ \ln(r/R) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_j}{r}\right)^m [\cos m(\theta - \theta_j)] \right] , & r > r_j \\ -\frac{e'}{2\pi\epsilon_o} \left[ \ln(r_j/R) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r}{r_j}\right)^m \cos [m(\theta - \theta_j)] \right] , & r < r_j \end{cases} \quad (3.6)$$

where  $R$  is the radius of some conducting shell concentrically surrounding the line charge which permits the potential to go to zero as the radial variable approaches  $R$ .  $R$  is large compared to the source radial dimension. We further assume that there are  $n_1$  electrons having source coordinate greater than the field coordinate ( $r_j > r$ ) and  $n_2$  with source coordinate less than the field coordinate ( $r_j < r$ ). The total scalar potential may be expressed by

$$\begin{aligned} \phi(r, \theta; r_j, \theta_j) = & -\frac{1}{2\pi\epsilon_o} \left[ \lambda \ln(r/R) + e' \sum_{j=1}^{n_1} \ln(r_j/R) - \right. \\ & \left. - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r}{r_j}\right)^m \cos [m(\theta - \theta_j)] \right] + \\ & + n_2 e' \ln(r/R) - e' \sum_{j=1}^{n_2} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_j}{r}\right)^m \cos [m(\theta - \theta_j)] \end{aligned} \quad (3.7)$$

The vector potential is treated in a manner analogous to the scalar potential. It satisfies a vector Poisson's equation whose solution may be expressed in terms of Green's dyadic.

Presenting just the results we have

$$\begin{aligned} \vec{A}(\vec{r}, \theta) = & -\frac{\mu_0}{2\pi} \left[ n_0 e v_0 \ln(r/R) \vec{e}_z + \right. \\ & + e' \sum_{j=1}^{n_1} \vec{v}_j \left\{ \ln(r_j/R) - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{r}{r_j} \right)^m \cos [m(\theta - \theta_j)] \right\} \\ & \left. + e' \ln(r/R) \sum_{j=1}^{n_2} \vec{v}_j \left\{ 1 - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{r_j}{r} \right)^m \cos [m(\theta - \theta_j)] \right\} \right] \end{aligned} \quad (3.8)$$

where  $n_0$  represents the number of electrons per unit length in the beam, i.e.  $dn/dz$  and  $v_0$  is the velocity of the beam electrons. The coordinates  $r_j$  and  $\theta_j$  in eqs. (3.6) and (3.7) and the velocity vector  $\vec{v}_j$  are randomly distributed so we shall replace them by their expectation values, through the mean fields approximations (so called because the vector and scalar potential fields are consequently replaced by their average values)

$$\begin{aligned} \sum_{j=1}^{n_1} f(\vec{v}_j) &= n_1 \int P(\vec{v}_j) f(\vec{v}_j) d\vec{v}_j / \int P(\vec{v}_j) d\vec{v}_j \\ \sum_{j=1}^{n_k} f(\vec{r}_j) &= n_k \int P(\vec{r}_j) f(\vec{r}_j) d\vec{r}_j / \int P(\vec{r}_j) d\vec{r}_j \\ \sum_{j=1}^{n_h} f(\vec{v}_j, \vec{r}_j) &= n_h \int P(\vec{v}_j, \vec{r}_j) f(\vec{v}_j, \vec{r}_j) d\vec{v}_j d\vec{r}_j / \int P(\vec{v}_j, \vec{r}_j) d\vec{v}_j d\vec{r}_j \end{aligned} \quad (3.9)$$

In the notation of eq. (3.9)  $P$  is the probability distribution function and  $f$  may be any arbitrary function. In this mean field approximation, the average vector potential momentum  $\langle e\vec{A} \rangle$  is assumed small compared to the average momentum  $\langle \vec{p} \rangle$  for any electron in the plasma. This zero order approximation will permit us to calculate dynamical quantity averages including vector potential effects at a later stage. The initial distribution function we shall use is

$$P_0(\vec{r}_j, \vec{v}_j) = \exp \{ -\beta [H_0(\vec{v}_j) + e\phi_0(\vec{r}_j)] \} \quad (3.10)$$

$H_0$  is the kinetic energy of the  $j$ th electron,  $e\phi_0$  is its potential energy with respect to a shielded line charge potential. For a line charge of radius on the order of 1 cm, surrounded by a plasma of electrons having a Debye wave number  $k_D$  on the order of  $10^5 \text{ cm}^{-1}$  we may approximate the Debye Hückel potential for a system possessing cylindrical and azimuthal symmetry. The exact solution is derived in Appendix A. Quoting the result,

$$\phi_0(r) = \frac{1}{2\pi} \frac{\lambda}{\epsilon_0} K_0(k_D r) \quad (3.11)$$

$K_0$  is a modified Bessel function and is related to the ordinary Bessel function through the relation

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \quad (3.12)$$

Since the product of Debye wave number and radial coordinate for our system is always much greater than unity we approximate eq.

(3.12) with the expression

$$\phi_o(r) \approx \frac{\lambda}{\epsilon_o \sqrt{8\pi k_D}} \left( e^{-k_D r / \sqrt{r}} \right) \quad (3.13)$$

Calculating the average vector and scalar fields using eqs. (3.7), (3.8), (3.9), (3.10) and (3.13) requires our evaluating  $\langle \ln(r_j/R) \rangle$ ,  $\langle (r/r_j)^m \rangle$ ,  $\langle (r_j/r)^m \rangle$ , and  $\langle \vec{v}_j \rangle$ . Because the Debye-Huckel potential of eqs. (3.11) or (3.13) is central in nature, i.e.

$$\phi_o(\vec{r}) = \phi_o(r)$$

we find

$$\langle (r/r_j)^m \rangle = \langle (r_j/r)^m \rangle = 0 \quad (3.14)$$

In the absence of the vector potential, there is no preferred direction in velocity space and we have

$$\langle \vec{v}_j \rangle = 0 \quad (3.15)$$

The only non-vanishing average is the logarithm term.

$$\langle \ln(r_j/R) \rangle = - (R-R_o)^{-1} \left[ R_o \ln\left(\frac{R_o}{R}\right) + (R-R_o) + \beta \delta I \right] \quad (3.16)$$

$$I = (k_D R)^{-1} \left\{ R_o^{\frac{1}{2}} e^{-k_D R_o} - R^{\frac{1}{2}} e^{-k_D R} \right\} - \quad (3.17)$$

$$- k_D^{-1} \left\{ R_o^{-\frac{1}{2}} e^{-k_D R_o} - R^{-\frac{1}{2}} e^{-k_D R} \right\}$$

$$\delta = e\lambda/\epsilon_o \sqrt{8\pi k_D} \quad (3.18)$$

and  $R_0$  is the vanishingly small but finite radius of the line source. Substituting the results of eqs. (3.14) through (3.16) into eqs. (3.7) and (3.8) gives the average vector and scalar potentials that shall be used in the Hamiltonian of eq. (3.1)

$$\begin{aligned}
 H = & \frac{1}{2m} \sum_{j=1}^N \left[ p_j^2 + \frac{\mu_0 n_0 v_0 e^2}{\pi} \ln\left(\frac{r_j}{R}\right) p_{zj} + \right. \\
 & \left. + \left( \frac{\mu_0 n_0 v_0 e^2}{2\pi} \right)^2 \ln^2\left(\frac{r_j}{R}\right) \right] - \\
 & - \frac{e}{2\pi\epsilon_0} \sum_{j=1}^N \left( \lambda + n_2(r_j) e' \ln\left(\frac{r_j}{R}\right) - n_1(r_j) N e' (R - R_0)^{-1} \right. \\
 & \left. \times \left[ R_0 \ln\left(\frac{R_0}{R}\right) + (R - R_0) + \beta \delta I \right] \right)
 \end{aligned} \tag{3.19}$$

The number of electrons  $n_1$  and  $n_2$  that enter into (3.19) depends on the radial coordinate  $r_j$ . Using the distribution function of eq. (3.10) and the fact that  $e\phi_0/kT \ll 1$ , we find that the electrons for all practical purposes are uniformly distributed throughout the medium, and we obtain

$$\begin{aligned}
 n_1(r_j) & \sim n_p \pi L_0 (R^2 - r_j^2) \\
 n_2(r_j) & \sim n_p \pi L_0 (r_j^2 - R_0^2)
 \end{aligned} \tag{3.20}$$

where  $n_p$  is the number density of electrons in the plasma, and  $L_0$  is an arbitrary length in the  $z$  direction. The average number of electrons is given by

$$\langle n_1(r_j) \rangle = \langle n_2(r_j) \rangle \sim n_p / 2 \tag{3.21}$$

## IV. THE LIOUVILLE OPERATOR AND THE TIME PROPAGATOR

The Liouville operator for a system of N electrons is given by <sup>12</sup>

$$L = -i \sum_{j=1}^N \left( \frac{\partial H}{\partial \vec{p}_j} \frac{\partial}{\partial \vec{r}_j} - \frac{\partial H}{\partial \vec{r}_j} \frac{\partial}{\partial \vec{p}_j} \right) \quad (4.1)$$

In a cylindrical coordinate representation with azimuthal symmetry this becomes

$$L = -i \sum_{j=1}^N \left( \frac{\partial H}{\partial p_{jr}} \frac{\partial}{\partial r_j} - \frac{\partial H}{\partial r_j} \frac{\partial}{\partial p_{jr}} \right) \quad (4.2)$$

From the Hamiltonian of equation (3.19) we immediately find

$$\frac{\partial H}{\partial p_{jr}} = \frac{p_{jr}}{m} \quad (4.3)$$

$$\frac{\partial H}{\partial r_j} = \left[ K_1 \ln\left(\frac{r_j}{R}\right) + K_2 p_{jz} + K_3 \right] \left(\frac{1}{r_j}\right) \quad (4.4)$$

$$K_1 = \frac{\mu_o^2 n_o^2 v_o^2 e^4}{2\pi^2} \quad (4.5)$$

$$K_2 = \frac{\mu_o n_o v_o e^2}{\pi m} \quad (4.6)$$

$$K_3 = -\frac{e\lambda}{2\pi\epsilon_o} \quad (4.7)$$

Substitution of eqs. (4.3) and (4.4) into eq. (4.2) leads to an explicit expression for the Liouville operator

$$L = -i \sum_{j=1}^N \left( \frac{p_{jr}}{m} \frac{\partial}{\partial r_j} - \left[ K_1 \ln \left( \frac{r_j}{R} \right) + K_2 p_{jz} + K_3 \right] \frac{1}{r_j} \frac{\partial}{\partial p_{jr}} \right) \quad (4.8)$$

For any function  $\chi$  of momenta and coordinates,  $\vec{p}$  and  $\vec{q}$ , which does not explicitly depend on time,

$$\chi(\vec{p}(t), \vec{q}(t)) = \exp(tiL) \chi(\vec{p}(t_0), \vec{q}(t_0)) \quad (4.9)$$

and  $L$  depends on time implicitly through  $\vec{p}$  and  $\vec{q}$ .<sup>12</sup> It should be noted that  $L = L(\vec{p}(t_0), \vec{q}(t_0))$ , that is, the Liouville operator is expressed in terms of momenta and position vector at some arbitrary initial moment,  $t_0$ . The operator  $\exp(tiL)$  is known as a time propagator for it "propagates" a function at some initial arbitrary instant into a later time, according to eq. (4.9).

Using the power series expansion

$$\exp(tiL) = 1 + tiL + \frac{t^2}{2}(iL)(iL) + \dots \quad (4.10)$$

allows us to obtain results, repeating the application of  $iL$  to an arbitrary function of a dynamical variable as many times as is necessary.

## V. THE ELECTRIC CONDUCTIVITY TENSOR

If we explicitly utilize the propagator of eq. (4.10) in the conductivity tensor of eq. (2.7) we find

$$\sigma_{\mu\nu}(\omega) = \frac{ne^2}{m^2 kT} \lim_{\epsilon \rightarrow 0} \int_0^\infty dt e^{-(\epsilon+i\omega)t} \times \quad (5.1)$$

$$\times \langle \exp(itL) p_\mu(0) p_\nu(0) \rangle_0$$

Performing the indicated operations of time propagation, ensemble averaging, and Fourier transformation we find the components to be

$$\sigma_{rr}(\omega) = \frac{ne^2}{m} \left[ \frac{2}{i\omega} - \alpha_2 R \left( \frac{\pi kT}{2m} \right)^{\frac{1}{2}} \times \right. \quad (5.2)$$

$$\left. \times \frac{(1 - e^{-\alpha_2 R})}{\{(R - \alpha_2^{-1}) + \alpha_2^{-1} e^{-\alpha_2 R}\}} \frac{1}{\omega^2} \right]$$

$$\sigma_{zz}(\omega) = \frac{ne^2}{m} \left( \frac{1}{i\omega} \right) \left[ 1 - \frac{\alpha_1^2}{4\pi\alpha_2 R^2 m kT} \times \right. \quad (5.3)$$

$$\times \left\{ R^2 e^{-\alpha_2 R} + 2R\alpha_2^{-1} (2e^{-\alpha_2 R} + 1) + 6\alpha_2^{-2} (e^{-\alpha_2 R} - 1) \right\} /$$

$$/ (R - \alpha_2^{-1} + \alpha_2^{-1} e^{-\alpha_2 R}) ]$$

$$\begin{aligned}
\sigma_{rz}(\omega) = & \frac{ne^2}{m^2 kT} \left[ (2\pi mkT)^{\frac{1}{2}} \frac{\alpha_1}{4\pi R} \left\{ R\alpha_2^{-1} (e^{-\alpha_2 R} + 1) + \right. \right. \\
& + 2\alpha_2^{-2} (e^{-\alpha_2 R} - 1) \left. \right\} \frac{1}{i\omega} - \frac{\alpha_1 kT}{\pi} \left\{ 1 - \left( 1 + \frac{\alpha_2 R}{2} \right) e^{-\alpha_2 R} + \right. \\
& + \left. \frac{1}{2} (1 - e^{-\alpha_2 R}) \right\} \frac{1}{\omega^2} \left. \right] / \left( R - \alpha_2^{-1} + \alpha_2^{-1} e^{-\alpha_2 R} \right) \\
= & \sigma_{zr}^*(\omega)
\end{aligned} \tag{5.4}$$

where

$$\alpha_1 = \mu_0 n_0 v_0 e^2$$

$$\alpha_2 = e(\lambda + n_2 e') / 2\pi \epsilon_0 R kT$$

and  $R_0$ , the radius of the line source is ignored compared to  $R$ , the radius of the outer conductor. Examining the leading terms of the conductivity diagonal elements we see they are what one would expect using Drude theory. The remaining terms involve higher order inverse powers of frequency, and arise from the interaction of plasma electrons with themselves and with the external line source. These components diverge in the low frequency limit since we ignored binary collisions, which serve as a damping mechanism for the system. Thus the resistivity of such a system in an applied DC electric field should be zero, and its conductivity, infinite.

The  $z$  and  $r$  components of current that flow in the plasma are given by a three dimensional extension of eq. (2.5).

The response function  $\hat{\sigma}_{\mu\nu}(t)$  is found from the Fourier transform of the conductivity. The external electric field may be written as in eq. (2.2). Performing the integration in eq. (2.5) yields a term that is harmonic in time, plus higher order terms that involve a product of various powers of time with  $\exp(i\omega t)$ . Consequently the current increases and oscillates in time and will increase indefinitely without approaching a stationary state. This result differs from that described by Balescu<sup>20</sup> for a homogeneous plasma in an externally applied field. He calculated the response current from a Green's function solution to the Boltzmann equation and arrived at a stationary state. However Balescu has accounted for the potential energy of interaction specifically through an iterative perturbation solution for the Fourier transformed Green's function. We have used a mean field approximation to arrive at a time propagator which then leads to dynamical quantities that monotonically increase with time, never reaching a steady-state value.

## VI. CONCLUSIONS

The correlation function approach to the electric conduction problem of a beam-plasma system is successful in predicting the frequency dependent conductivity tensor in a mean-field approximation. The local behavior of the conductivity is ignored in this calculation. Such an effect will be considered in a future paper. The effects of electron-electron, electron-ion, and electron-neutral binary collisions have also been ignored and it is hoped that this too shall be studied at a later date. The general behavior of the conductivity components follows the classical Drude theory<sup>1,6</sup> results with the interaction between beam and plasma showing up as higher order terms in inverse powers of frequency. The off-diagonal component does not possess Drude-like behavior as its leading term, however. Thus the response current in the plasma increases with time indefinitely, and a steady-state or purely oscillating current is not obtained within this model.

## APPENDIX A. THE SHIELDED LINE SOURCE POTENTIAL

According to the Debye-Hückel theory as presented by Jackson<sup>6</sup> we imagine the line charge surrounded by a specified distribution of electrons with a uniform background of positive charge. The distribution function for the electrons will be the canonical one described in eq. (3.10). Then Poisson's equation is written

$$\nabla^2 \phi_0(r) = \frac{\lambda}{\epsilon_0} \frac{\delta(r)}{r} - \frac{n_0 e}{\epsilon_0} \left[ e^{-\beta e \phi_0} - 1 \right] \quad (\text{A.1})$$

The first terms on the right hand side is the charge distribution of the external source, the second represents the electron plus uniform ion distributions.  $\beta$  is just the Boltzmann factor  $(kT)^{-1}$ . For a plasma having a temperature of a few thousand degrees with a Debye wave number of

$k_D \sim 10^5 \text{ cm}^{-1}$ , the condition

$$e\phi_0(r)/kT \ll 1$$

is easily met. So we may linearize eq. (A.1)

$$\frac{d}{dr} \left( r \frac{d\phi_0}{dr} \right) - r k_D^2 \phi_0 = - \frac{\lambda}{\epsilon_0} \delta(r) \quad (\text{A.2})$$

where

$$k_D^2 = n_0 e^2 \beta / \epsilon_0 \quad (\text{A.3})$$

is the Debye wave number. We have assumed cylindrical symmetry to arrive at eq. (A.2). Its solution is found in Morse and Feshbach.<sup>21</sup>

$$\phi_0(r) = \frac{1}{2\pi} \left( \frac{\lambda}{\epsilon_0} \right) K_0(k_D r) \quad (\text{A.4})$$

APPENDIX B. AVERAGE VALUE OF THE  
LOGARITHMIC TERM IN THE MEAN FIELD APPROXIMATION

The average value of the logarithmic term in the expressions for the scalar and vector potential of eqs. (3.7) and (3.8) is found from eqs. (3.9) and (3.13).

$$\begin{aligned} \langle \ln(r_j/R) \rangle = & \int_{R_0}^R \exp \left\{ -\beta \left[ \delta \exp(-k_D r_j) / r_j^{\frac{1}{2}} \right] \right\} \times \\ & \times \ln(r_j/R) dr_j / \int_{R_0}^R \exp \left\{ -\beta \left[ \delta \exp(-k_D r_j) / r_j^{\frac{1}{2}} \right] \right\} dr_j \end{aligned} \quad (B.1)$$

The limits of integration reflect the upper and lower bounds on the radial coordinate,  $R$  representing the radius of the outer conducting shell, and  $R_0$  the small, but finite beam radius.

Inspection of the exponential argument reveals that a linearization is a suitable approximation for the typical beam-plasma systems that are of interest to us, i.e.

$$\text{MAX} |e\phi_0(r_j)| / kT \ll 1$$

Retaining the linear first order term in the expansion of the numerator and unity for the denominator in eq. (B.1) yields

$$\langle \ln(r_j/R) \rangle \approx - (R-R_0)^{-1} \left\{ R_0 \ln(R_0/R) + (R-R_0) + \beta \delta I \right\} \quad (B.2)$$

where

$$I \equiv \int_{R_0}^R \exp \{ -k_D r_j \} \ln(r_j/R) / r_j^{\frac{1}{2}} dr_j \quad (B.3)$$

The integral in eq. (B.3) may be expressed as a combination of incomplete gamma functions if we use the leading term in a series

approximation for the logarithm. Then

$$\ln x \approx x - 1, \quad 0 < x \leq 2$$

We find

$$\begin{aligned} I = R^{\frac{1}{2}} & \left[ (k_D R)^{-3/2} \gamma\left(\frac{3}{2}, k_D R\right) - (k_D R)^{-\frac{1}{2}} \gamma\left(\frac{1}{2}, k_D R\right) \right] - \\ & - R^{\frac{1}{2}} \left[ (k_D R)^{-3/2} \gamma\left(\frac{3}{2}, k_D R_0\right) - (k_D R)^{-\frac{1}{2}} \gamma\left(\frac{1}{2}, k_D R_0\right) \right] \end{aligned} \quad (\text{B.4})$$

The incomplete gamma function,  $\gamma$  is defined in terms of the ordinary gamma function  $\Gamma(\zeta)$  and the complementary incomplete gamma function  $\Gamma(\zeta, \xi)$

$$\Gamma(\zeta) = \Gamma(\zeta, \xi) + \gamma(\zeta, \xi) \quad (\text{B.5})$$

$\Gamma(\zeta, \xi)$  has an asymptotic representation (since  $k_D r \gg 1$ )

$$\begin{aligned} \Gamma(\zeta, \xi) = \xi^{\zeta-1} e^{-\xi} \sum_{m=0}^{M-1} \frac{(-1)^m \Gamma(1-\zeta+m)}{\xi^m \Gamma(1-\zeta)} + \\ + O(|\xi|^{-M}) \end{aligned} \quad (\text{B.6})$$

for  $|\xi| \rightarrow \infty$

Consequently we obtain

$$\gamma(\zeta, \xi) = \Gamma(\zeta) - \xi^{\zeta-1} e^{-\xi} \quad (\text{B.7})$$

where we have retained the first term in the asymptotic series for  $\Gamma(\zeta, \xi)$ . Substitution of eq. (B.7) into eq. (B.4) yields eq. (3.17).

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